Thickness Corrections for Capacitive Obstacles and Strip Conductors*

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Summary—**Capacitive thickness corrections are derived exactly for two basic geometries involving pairs of semi-infinite plates. In one arrangement the pair of plates are coplanar, while in the other they are parallel to each other. In each case the total capacitance per unit length between the pair of plates is infinite, but the incremental increase of capacitance when the thickness is increased from zero to a value t is finite. These capacitance increments are evaluated, and it is shown how they may be used as approximate thickness corrections in a great varity of more complicated geometries involving capacitive obstacles in waveguide, coaxial line, and artificial dielectric media. They may also be applied to coupled-strip-line conductors. As examples, the corrections are applied in detail to a waveguide iris, and to three useful coupled-strip-line cross sections.**

I. Introduction

SOLUTIONS for many capacitive-obstacle and strip-transmission-line cross sections are available only in the case of zero-thickness conductors. If the conductors are sufficiently thin, as in the usual photo-etched strip-line circuit, the accuracy of the solution will be good, but in numerous practical microwave structures finite thickness has a large effect. Except in a few simple cases, exact solutions for the parameters of cross sections containing thick edges are very difficult to achieve. In order to alleviate this problem, two basic thick-edge cross sections are considered in this paper, and the increase in capacitance due to finite thickness is given for each case. These capacitance increments may be applied as corrections to a wide variety of practical structures for which the zero-thickness solutions are either available or obtainable.

The two basic cross sections treated in this paper are shown in Fig. 1. One consists of a pair of semi-infinite thick plates in a coplanar arrangement [Fig. 1(a)], while the other consists of a pair of semi-infinite thick plates in a parallel arrangement [Fig. 1(c)]. In each case, an electric wall may be inserted in the plane of symmetry, thus creating the unsymmetrical equivalents shown in Fig. 1(b) and 1(d). The total capacitance between the semi-infinite conductors are $C_1'(t/s)$ and $C_2'(t/s)$ farads per unit length, respectively, in which the *prime* mark indicates that the capacitance is computed for one unit of length of the cylindrical conductors. Because the plates are semi-infinite, $C_1'(t/s)$ and $C_2'(t/s)$ are infinite. However, the incremental increase in capacitance when t is increased from zero with s held constant is a finite quantity. This increment of capacitance is defined in each case as follows:

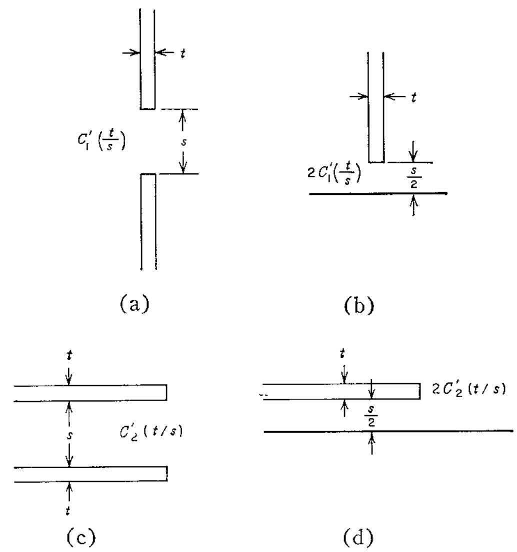

Fig. 1—Basic semi-infinite plate cross sections. (a) and (b) Coplanar arrangement and unsymmetrical equivalent. (c) and (d) Parallel arrangement and unsymmetrical equivalent.

$$\Delta C_{1,2}'(t/s) = C_{1,2}'(t/s) - C_{1,2}'(0). \qquad (1)$$

If one considers the field-distribution patterns for $t=0$ and for $t>0$, one realizes that they differ mainly near the plate edges, with negligible difference far from the edges. Thus the increments $\Delta C_1'(t/s)$ and $\Delta C_2'(t/s)$ arise from field distortions close to the edges, and their values would be affected only very slightly if additional structural surfaces were present at locations relatively distant. Consequently the capacitance increments derived for Fig. 1(a) and 1(c) may be used as thickness corrections in the case of more complex cross sections for which zero-thickness solutions can be obtained.

Fig. 2 shows a number of practical configurations to which the capacitive increment for the coplanar-plate case may be applied as a thickness correction. Solutions valid for zero thickness are already available for most of these,[1–4] and are feasible to obtain for the others. In the coaxial-line cases, the total correction is $2\pi r_e \cdot 2\Delta C_1'(t/s)$, where r_e is a judiciously chosen radius between that of

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[1] N. Marcuvitz, "Waveguide Handbook," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 218–221; 1951.
[2] *Ibid.*, pp. 229–238.
[3] S. B. Cohn, "Shielded coupled-strip transmission line," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 29–38; October, 1955.
[4] S. B. Cohn, "Analysis of the metal-strip delay structure for microwave lenses," *J. Appl. Phys.*, vol. 20, pp. 251–262; March, 1949. Also, "Addendum," p. 1011; October, 1949.

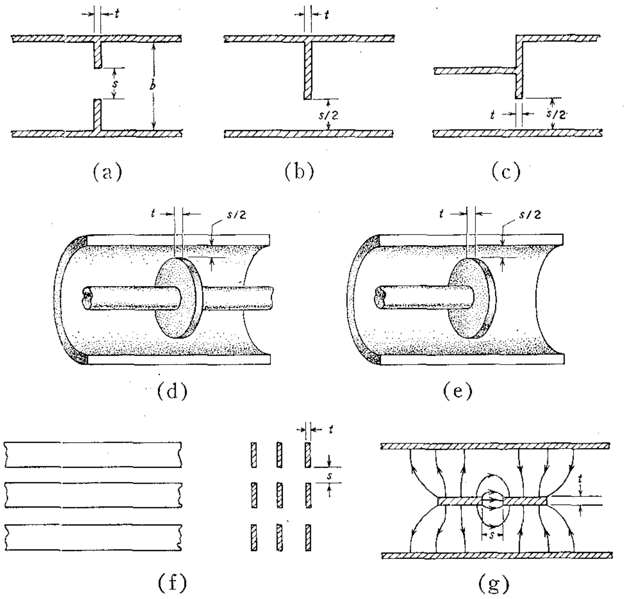

Fig. 2—Practical cases where the coplanar plate thickness correction may be applied. (a)–(e) Examples of capacitive obstacles of moderate thickness in waveguide and coaxial line. (f) Metal strip artificial dielectric medium. (g) Coplanar coupled strip line in odd mode.

the disk and that of the outer conductor. In order for the correction to apply accurately to the cases in Fig. 2, it is necessary that s/λ_g and s/r be sufficiently small compared to unity (λ_g is the guide wavelength and r is the distance from the center point of the symmetrical configuration to the nearest extraneous surface). Fortunately, these requirements are not very stringent. On the basis of a study of the waveguide iris of Fig. 2(a) (see Section IV), it is believed that the correction will usually yield good results for s/λ_g as large as 1/4, and s/r at least as large as 1/2.

Fig. 3 shows situations in which the capacitive increment for the parallel-plate case may be applied. Formulas for the odd- and even-mode characteristic impedances of the coupled-strip lines in Fig. 3(a) and 3(b) have recently been given in another paper, assuming zero thickness.[5] As shown in Section V, the capacitive increment for the parallel-plate case may be used to compute a thickness correction for the odd mode. Fig. 3(c) shows a strip conductor over a ground plane, for which the same correction term is applicable. Fig. 3(d) shows an example of a coaxial cavity containing a capacitive disk closely spaced from an end wall. Here, also, an effective radius r_e must be judiciously chosen, yielding a total thickness correction $\Delta C = 2\pi r_e \cdot 2\Delta C_2'(t/s)$. This problem has been considered in detail by the author, and has resulted in formulas for the resonant frequency and unloaded Q of coaxial cavities of the type shown in Fig. 3(d). The minimum allowable distance from the edges to the nearest extraneous surface in the configurations of Fig. 3 cannot be determined

[5] S. B. Cohn, "Characteristic impedances of broadside-coupled strip transmission lines," IRE Trans. on Microwave Theory and Techniques, this issue, p. 633.

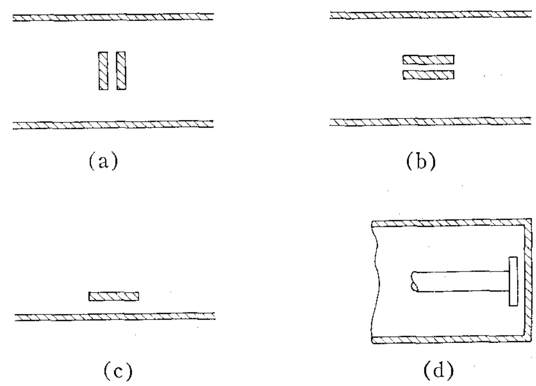

Fig. 3—Practical cases where the parallel-plate thickness correction may be applied.

precisely. However, it is expected that good results will usually be obtained for $(s+2t)/r$ as large as 1/2, and for $(s+2t)/\lambda_g$ as large as 1/4.

II. Thickness-Correction Term for Coplanar Plates

As derived in Appendix I, the incremental capacitance per unit length is as follows for the coplanar plates of Fig. 1(a):

$$\Delta C_1'(t/s) = \frac{2\epsilon}{\pi} \ln \left[\frac{E(k) - (1/2)k'^2 K(k)}{\sqrt{k}} \right] \tag{2}$$

where the parameters k and k' are solved from the following equations as functions of t/s:

$$\frac{t}{s} = \frac{\dfrac{1+k^2}{2} K(k') - E(k')}{2\left[E(k) - \dfrac{k'^2}{2} K(k) \right]} \tag{3}$$

$$k' = \sqrt{1-k^2}. \tag{4}$$

The units of $\Delta C_1'(t/s)$ are the same as those of the permittivity ϵ, which for free space has the value 8.85 $(10)^{-12}$ farads per meter, 0.0885 $\mu\mu$f per cm, or 0.225 $\mu\mu$f per inch. $K(k)$ and $E(k)$ are complete elliptic integrals of the first and second kinds, respectively.

Eqs. (2)–(4) have been computed, and the resulting curve is given in Fig. 4. The procedure used was first to select values of k, and then to calculate the corresponding values of $\Delta C_1'$ and t/s. The quantity plotted is $[\Delta C_1'(t/s) - \epsilon t/s]/\epsilon$, in which the term $\epsilon t/s$ is the parallel-plate capacitance per unit length neglecting fringing. By subtracting this parallel-plate term from $\Delta C_1'$, the curve is made to approach a constant value as t/s is made large enough so that the fringing fields on the two sides of the configuration become independent. It is seen that this occurs for $t/s \geq 1$. The limiting value of $\Delta C_1'$ for $t/s \geq 1$ is of interest. With the aid of limiting values of the elliptic integrals, one may show this to be

$$\frac{\Delta C_1'}{\epsilon} = \frac{t}{s} + \frac{2}{\pi}\left(1 + \ln\frac{\pi}{8}\right) = \frac{t}{s} + 0.0415. \quad (5)$$

The curve approaches zero as t/s goes to zero. For $t/s < 0.001$ the correction is very small, so that parameter values for zero-thickness cases may be used with high accuracy. For $t/s > 0.5$, the correction is essentially constant, so that data for isolated step discontinuities may be applied accurately. The range $0.001 < t/s < 0.5$ is, therefore, the region of interest for which the thickness correction presented here is needed. Most practical capacitive-iris structures fall within this range.

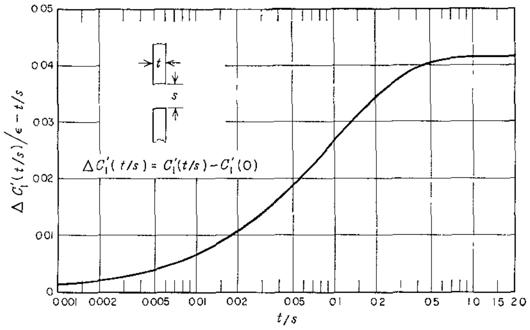

Fig. 4—Plot of coplanar capacitance correction.

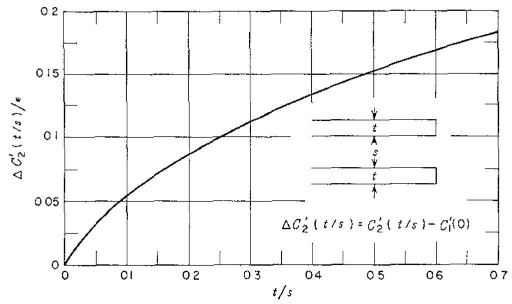

Fig. 5—Plot of parallel-plate capacitance correction.

III. Thickness-Correction Term for Parallel Plates

The following formula is derived in Appendix II for the incremental capacitance per unit length of the parallel plates of Fig. 1(c).

$$\Delta C_2'(t/s) = \frac{\epsilon}{2\pi}\left[\left(1 + \frac{t}{s}\right)\ln\left(1 + \frac{t}{s}\right) - \frac{t}{s}\ln\frac{t}{s}\right]. \quad (6)$$

Eq. (6) is plotted in Fig. 5.

IV. Example of Thick Capacitive Iris in Waveguide

The manner in which the thickness correction may be applied to practical configurations will be illustrated by the example of a thick capacitive iris in waveguide. (This case has also been treated by Marcuvitz by different methods.[6]) Fig. 6(a) shows a longitudinal E-plane section through a rectangular waveguide containing a zero-thickness, perfectly conducting, symmetrical capacitive iris. It is assumed that only the TE_{10} mode is propagating. In the limiting case of $b/\lambda_g \to 0$, the equivalent normalized susceptance of the iris[1] is

$$\frac{B}{Y_0} = \frac{4b}{\lambda_g}\ln\csc\left(\frac{\pi s}{2b}\right), \quad (7)$$

where B is the susceptance of the iris, Y_0 is the characteristic admittance of the waveguide, λ_g is the guide wavelength, and b and s are dimensions defined in Fig. 6(a).

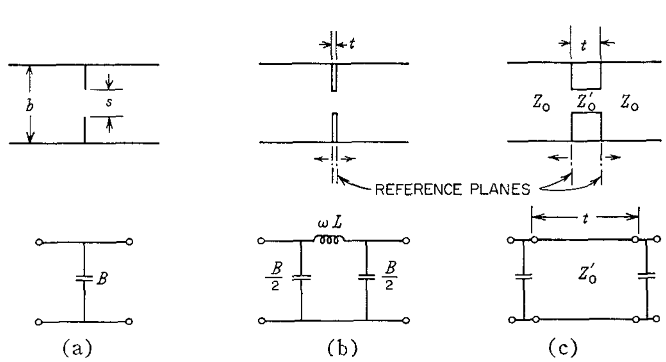

Fig. 6—Capacitive obstacle in rectangular waveguide for (a) very thin, (b) moderately thick, and (c) very thick cases.

In applying the capacitive thickness correction, it is necessary to employ an equivalence theorem between E-plane structures in TE_{10}-mode rectangular waveguide and in TEM-mode parallel-plane transmission line. If the structure has the same boundary in all longitudinal E-plane sections in both cases, then the normalized element values of the waveguide equivalent circuit are identical to those of the parallel-plane equivalent circuit, if λ_g of the TE_{10} mode is used in place of λ of the TEM mode.[7] The normalized susceptance increment due to thickness may therefore be evaluated in terms of $\Delta C_1'$ as follows. Assume a parallel-plane TEM-mode transmission line with E-plane dimension b and H-plane dimension a. The characteristic admittance is $Y_0 = (a/b)\sqrt{\epsilon/\mu}$ mhos, where ϵ and μ are the permittivity and permeability of the filling medium in mks units. The susceptance increment is $\Delta B = a\omega\Delta C_1' = 2\pi a\Delta C_1'/\lambda\sqrt{\epsilon\mu}$. Hence, the normalized susceptance increment due to thickness is

$$\frac{\Delta B}{Y_0} = \frac{2\pi b}{\lambda}\cdot\frac{\Delta C_1'}{\epsilon}.$$

[6] Marcuvitz, *op. cit.*, pp. 248–255 and 404–406.
[7] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," McGraw-Hill Book Co., Inc., New York, N. Y., p. 172; 1948.

By virtue of the equivalence theorem, the corresponding quantity in rectangular waveguide is

$$\frac{\Delta B}{Y_0} = \frac{2\pi b}{\lambda_g}\cdot\frac{\Delta C_1'}{\epsilon}, \tag{8}$$

where $\Delta C_1'/\epsilon$ is obtained as a function of t/s from (2)–(4), or from the graph in Fig. 5.

In addition to the increase in capacitive susceptance, thickness also requires a series inductive reactance to be added to the equivalent circuit, as shown in Fig. 6(b). This reactance represents the magnetic-field energy in the gap region of the thick iris. In the parallel-plane case, the inductance of this region is simply $L = \mu ts/a$, while the characteristic impedance is $Z_0 = \sqrt{\mu/\epsilon}(b/a)$. Therefore, the normalized reactance is $\omega L/Z_0 = 2\pi ts/\lambda b$. In waveguide, this becomes,

$$\frac{\omega L}{Z_0} = \frac{2\pi ts}{\lambda_g b}. \tag{9}$$

An interesting check on the thickness correction may be obtained in the very thick iris case of Fig. 6(c). If $t/\lambda_g \ll 1$ and $t/s \geq 1$, the total shunt susceptance is obtained by adding (7) and (8), with the aid of (5):

$$\frac{B}{Y_0} = \frac{4b}{\lambda_g}\left[\ln\csc\left(\frac{\pi s}{2b}\right) + \frac{\pi t}{2s} + 1 + \ln\frac{\pi}{8}\right]. \tag{10}$$

In the limit $s/b \to 0$, this reduces to

$$\frac{B}{Y_0} = \frac{4b}{\lambda_g}\left(\ln\frac{b}{4s} + 1 + \frac{\pi t}{2s}\right). \tag{11}$$

The total susceptance for this case may also be evaluated by adding the parallel-plate capacitance to two times the step discontinuity capacitance. The normalized shunt susceptance appropriate to the parallel-plate capacitance is $(2\pi t/\lambda_g)(b/s)$, while the normalized step discontinuity susceptance[8] in the limit $s/b \to 0$ is $(2b/\lambda_g)[\ln(b/4s)+1]$. The total shunt susceptance is, therefore,

$$\frac{B}{Y_0} = 2\left(\frac{2b}{\lambda_g}\right)\left(\ln\frac{b}{4s} + 1\right) + \frac{2\pi tb}{\lambda_g s}$$
$$= \frac{4b}{\lambda_g}\left(\ln\frac{b}{4s} + 1 + \frac{\pi}{2}\frac{t}{s}\right). \tag{12}$$

This agrees exactly with (11), which was obtained by adding the thickness correction to the zero-thickness value. It should be remembered that this exact agreement assumes $s/b \ll 1$. However, more detailed calculations have indicated close agreement for $s/b \leq 0.25$, and fair agreement for s/b at least as large as 0.6.

[8] *Ibid.*, pp. 307–310.

V. Examples of Thick Coupled Strips

A. Coplanar-Coupled Strip Line

The coplanar-coupled strip transmission line of Fig. 7 will now be considered. The odd- and even-mode characteristic impedances of one strip to ground are related to the odd- and even-mode capacitances per unit length of one strip to ground by, respectively,

$$Z_{oo} = \frac{\sqrt{\mu\epsilon}}{C_{oo}'} \quad \text{and} \quad Z_{oe} = \frac{\sqrt{\mu\epsilon}}{C_{oe}'}, \tag{13}$$

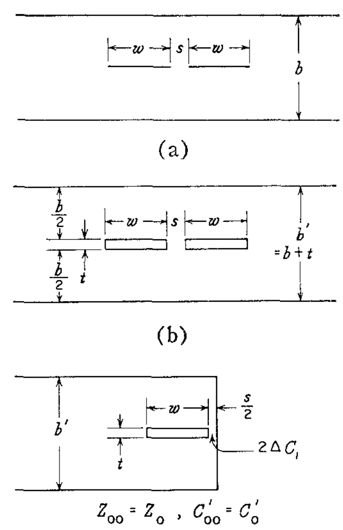

Fig. 7—Coplanar coupled-strip transmission line.

where μ and ϵ are the permeability and permittivity of the filling medium in mks units. First consider the odd mode. Symmetry permits a vertical electric wall to be inserted between the strips, reducing the cross section to that of Fig. 7(c). The characteristic impedance and capacitance per unit length of Fig. 7(c) are Z_{oo} and C_{oo}', the quantities desired. In general, C_{oo}' is a function of the dimension ratios w/b, s/b, and t/b. For $t/b = 0$ [Fig. 7(a)],

$$C_{oo}'\left(\frac{w}{b}, \frac{s}{b}, \frac{t}{b}\right) = C_{oo}'\left(\frac{w}{b}, \frac{s}{b}, 0\right), \tag{14}$$

which may be obtained exactly from Cohn.[3] For $t/b > 0$, b should be increased by t to $b' = b + t$, as shown in Fig. 7(b) and 7(c), so that the proportions above and below the strips will be unchanged. Then the thickness effect at the strip edges adjacent to the gap s requires the addition of $2\Delta C_1'(t/s)$ to $C_{oo}'(w/b, s/b, 0)$. The gap s is assumed to be small enough compared to $b/2$ and w for $\Delta C_1'(t/s)$ to apply accurately ($s < b/4$ and $s < w/2$ are probably sufficient). At the strip edges remote from the gap, the thickness effect requires the addition of $2C_f'(t/b') - 2C_f'(0)$, where $C_f'(t/b')$ is the fringing capacitance to ground per unit length from each corner of a semi-infinite plate of thickness t midway between infinite ground planes of spacing b', and $C_f'(0)$ is the same for $t=0$. Values of $C_f'(t/b')$ may be obtained from (5) or Fig. 3 of Cohn.[9] Hence for $t>0$,

$$C_{oo}'\left(\frac{w}{b'}, \frac{s}{b'}, \frac{t}{b'}\right) = C_{oo}'\left(\frac{w}{b}, \frac{s}{b}, 0\right) + 2\Delta C_1'(t/s) + 2C_f'(t/b') - 2C_f'(0). \quad (15)$$

For the even mode, the two strips are at the same potential, so that $\Delta C_1'(t/s)$ does not apply. Eq. (18) of Cohn[3] may be used to obtain a good approximation to Z_{oe} or C_{oe}'. Or, for s/b small, another very good approximation is

$$C_{oe}'\left(\frac{w}{b'}, \frac{s}{b'}, \frac{t}{b'}\right) = C_{oe}'\left(\frac{w}{b}, \frac{s}{b}, 0\right) + 2C_f'\left(\frac{t}{b'}\right) - 2C_f'(0), \quad (16)$$

where $C_{oe}'(w/b, s/b, 0)$ is the even-mode capacitance to ground per unit length for zero-thickness strips, obtainable exactly from Cohn.[3]

B. Broadside-Coupled Strip Line

Formulas valid for zero thickness have been obtained for the even- and odd-mode characteristic impedances of the two broadside-coupled strip cross sections of Figs. 8(a) and 9(a). The corresponding capacitances per unit length of one strip to ground, $C_{oo}'(w/b, s/b, 0)$ and $C_{oe}'(w/b, s/b, 0)$, are related to these characteristic impedances by (13).

First consider the odd mode in Fig. 8. For $t>0$, the proportions above and below the pair of strips may be preserved by increasing the plate spacing to $b'=b+2t$. The odd-mode potential distribution permits insertion of an electric wall at ground potential between the strips, reducing the cross section to that of Fig. 8(c). The characteristic impedance of the latter cross section is the quantity desired. This may be obtained from the capacitance per unit length of the odd mode, which is approximately as follows,

$$C_{oo}'\left(\frac{w}{b'}, \frac{s}{b'}, \frac{t}{s}\right) = C_{oo}'\left(\frac{w}{b}, \frac{s}{b}, 0\right) + 4\Delta C_2'\left(\frac{t}{s}\right). \quad (17)$$

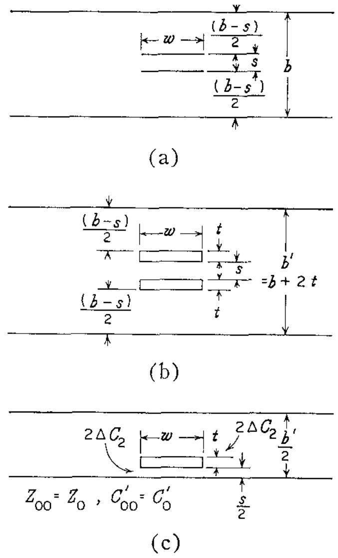

Fig. 8—Broadside coupled-strip transmission line; strips parallel to ground planes.

For the even mode, if $t \ll s$, the capacitance per unit length is,

$$C_{oe}'\left(\frac{w}{b'}, \frac{s}{b'}, \frac{t}{s}\right) = C_{oe}'\left(\frac{w}{b'}, \frac{s'}{b'}, 0\right), \quad (18)$$

in which $s'=s+2t$. If $t \gg s$, then

$$C_{oe}'\left(\frac{w}{b'}, \frac{s}{b'}, \frac{t}{s}\right) = \frac{1}{2} C_o'\left(\frac{w}{b'}, \frac{t'}{w}\right), \quad (19)$$

where $C_o'(w/b', t'/w)$ is the characteristic impedance of a single strip of width w and thickness t' midway between ground planes spaced by $b'=b+2t$.[9,10] Here, $t'=s+2t$. In the usual case of close coupling, (18) and (19) will agree very closely for any values of t and s.

Now consider Fig. 9. The odd-mode characteristic impedance of Fig. 9(b) is equal to the characteristic impedance of the reduced cross section shown in Fig. 9(c). The capacitance per unit length of the latter cross section is

$$C_{oo}'\left(\frac{w}{b}, \frac{s}{b}, \frac{t}{s}\right) = C'_{oo}\left(\frac{w}{b}, \frac{s}{b}, 0\right) + 4\Delta C_2'\left(\frac{t}{s}\right). \quad (20)$$

For the even mode, if $t \ll s$,

$$C_{oe}'\left(\frac{w}{b}, \frac{s}{b}, \frac{t}{s}\right) = C_{oe}'\left(\frac{w}{b}, \frac{s'}{b}, 0\right), \quad (21)$$

[9] S. B. Cohn, "Problems in strip transmission lines," IRE Trans. on Microwave Theory and Techniques, vol. MTT-3, pp. 119–126; March, 1955.

[10] R. H. T. Bates, "The characteristic impedance of the shielded slab line," IRE Trans. on Microwave Theory and Techniques, vol. MTT-4, pp. 28–33; January, 1956.

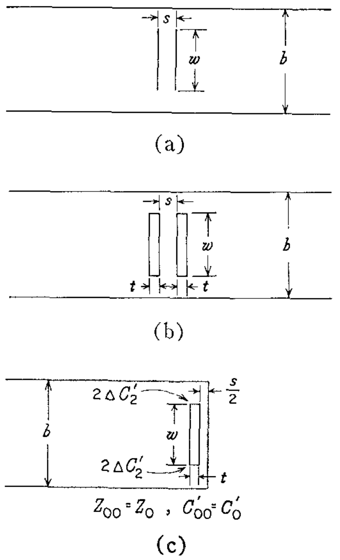

Fig. 9—Broadside coupled-strip transmission line; strips perpendicular to ground planes.

in which $s'=s+2t$. If $t \gg s$,

$$C_{oe}'\left(\frac{w}{b}, \frac{s}{b}, \frac{t}{s}\right) = \frac{1}{2} C_0\left(\frac{w'}{b}, \frac{t'}{w'}\right) \tag{22}$$

where $w'=s+2t$ and $t'=w$. Again, there will generally be little difference between (21) and (22) in the usual case of close coupling.

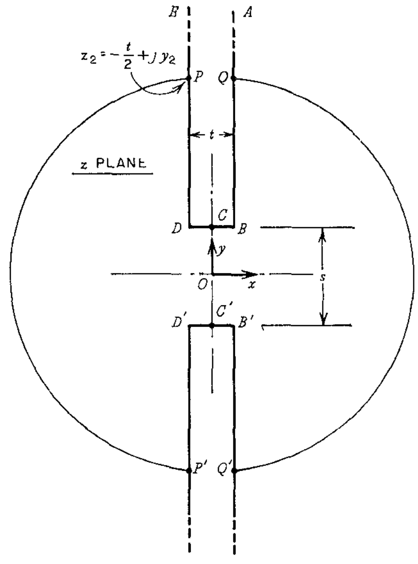

Fig. 10—Coplanar semi-infinite plate boundary in z plane.

where $k=1-k'^2$ is a real, positive parameter between 0 and 1 that may be solved as a function of t/s from the following equation:

$$\frac{t}{s} = \frac{K(k') - 2E(k') + k^2K(k')}{2[2E(k) - k'^2K(k)]}. \tag{24}$$

The symbol u is a complex variable that is a function of $z=x+jy$, and $sn(u, k)$ is an elliptic function.[11,12] The transformation relating z and u is

$$z = j\frac{s}{2}\left\{\frac{u\left[k'^2 - \dfrac{2E(k)}{K(k)}\right] - 2zn(u, k) - \dfrac{cn(u, k)dn(u, k)}{sn(u, k)}}{2E(k) - k'^2K(k)}\right\} - \frac{t}{2} \tag{25}$$

VI. Conclusions

Practical geometries containing thick capacitive obstacles or strips can seldom be solved rigorously, although solutions are frequently possible if zero thickness is assumed. The approximate thickness corrections derived in this paper may be applied to many practical cross sections for which zero-thickness solutions are available or can be obtained. The examples treated demonstrate the simplicity of this method.

Appendix I

Derivation of $\Delta C_1'(t/s)$

The derivation of (2) and (3) is based on a conformal-mapping analysis by Davy[11] of the double-semi-infinite-plate boundary shown in Fig. 10. He obtained the following formula for the capacitance per unit length between the two plates over the regions $QBCDP$ and $Q'B'C'D'P'$ of the boundary:

$$C_1' = -\frac{\epsilon}{\pi}\ln\left[k\, sn^2(u_2, k)\right], \tag{23}$$

where zn, cn, and dn also are elliptic functions.

At point P between D and E on the boundary, u is negative real, approaching zero as y_2 approaches infinity. The following limiting values are valid for $u \to 0$:

$$\begin{aligned} sn(u, k) &\to u \qquad & dn(u, k) &\to 1 \\ cn(u, k) &\to 1 \qquad & zn(u, k) &\to [1 - E(k)/K(k)]u. \end{aligned} \tag{26}$$

Substituting these in (25) we obtain in the limit $u \to 0$,

$$y_2 = \frac{s}{2[2E(k) - k'^2K(k)](-u_2)} \tag{27}$$

while (23) reduces to

$$C_1' = \frac{\epsilon}{\pi}\ln\left(\frac{1}{ku_2^2}\right). \tag{28}$$

[11] N. Davy, "The field between equal semi-infinite rectangular electrodes or magnetic pole pieces," *Phil. Mag.*, vol. 35, ser. 7, pp. 819–840; December, 1944.

[12] F. Oberhettinger and W. Magnus, "Anwendung der Elliptischen Funktionen in Physik und Technik," Springer-Verlag, Berlin Ger., 1949.

Combining (27) and (28) leads (in the limit $y_2 \to \infty$) to the following:

$$C_1'\left(\frac{t}{s}, \frac{y_2}{s}\right) = \frac{2\epsilon}{\pi} \ln \left[\frac{2y_2}{s} \cdot \frac{2E(k) - k'^2 K(k)}{\sqrt{k}} \right]. \quad (29)$$

This may be simplified for $t/s \to 0$, since then $k \to 1$, $k' \to 0$, $E(k) \to 1$, and $k'^2 K(k) \to 0$. Thus

$$C_1'\left(0, \frac{y_2}{s}\right) = \frac{2}{\pi} \ln \frac{4y_2}{s} \cdot \quad (30)$$

Now we shall define the capacitance increment $\Delta C_1'(t/s)$ by

$$\Delta C_1'(t/s) = \lim_{y_2 \to \infty} \left[C_1'\left(\frac{t}{s}, \frac{y_2}{s}\right) - C_1'\left(0, \frac{y_2}{s}\right) \right]. \quad (31)$$

Eq. (2) now follows directly from (29)–(31).

Appendix II

Derivation of $\Delta C_2'(t/s)$

The derivation of (6) will be outlined to illustrate the method of analysis, but with intermediate steps omitted for brevity.

E. Weber[13] gives the transformation $z = f(w)$ that relates the z and w planes in Fig. 11 such that all the conducting boundaries in the z plane are transformed into the real axis of the w plane. The top, bottom, and end of the semi-infinite plate in the z plane transform into the negative real axis of the w plane, while the infinite boundary at ground potential in the z plane transforms into the positive real axis of the w plane. A minute break in the real axis may be assumed at $u = 0$ to permit the discontinuity in potential.

The electric-field lines above and to the right of the plate in the z plane become semicircles as $|z|$ becomes large. Below the plate, the field lines become straight vertical lines as x recedes to minus infinity. In the w plane, all electric-field lines are semicircles with center at $w = 0$.

The capacitance per unit length in the w plane due to the electric flux contained between field lines of radius u_a and u_b is

$$C_{ab}' = \frac{\epsilon}{\pi} \ln \frac{u_b}{u_a} \cdot \quad (32)$$

[13] E. Weber, "Electromagnetic Fields, Vol. 1," John Wiley and Sons, Inc., New York, N. Y., p. 351, (55); 1950. (Note that p should be related to k by $p = 2k^2 - 1 + 2k\sqrt{k^2 - 1}$.)

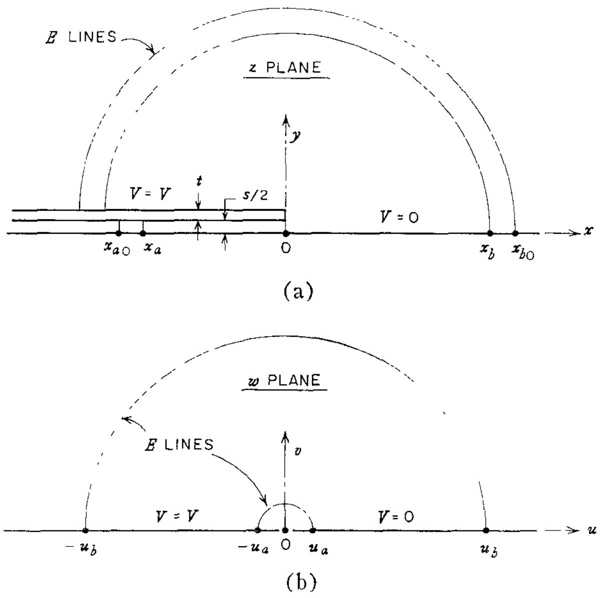

Fig. 11—Transformation between z and w planes used in derivation of $\Delta C_2'(t/s)$.

Let points $z_a = x_a + j0$ and $z_b = x_b + j0$ correspond to $w_a = u_a + j0$ and $w_b = u_b + j0$, respectively. Because of a basic property of conformal transformations, we know that the capacitance per unit length in the z plane due to the electric flux contained between field lines terminating at x_a and x_b is exactly equal to C_{ab}'. Now, if t is reduced to zero with s unchanged, points x_a and x_b move to x_{a0} and x_{b0}, with the capacitance per unit length still equal to the same value, C_{ab}'. But in comparing the $t = 0$ and $t > 0$ cases, we must evaluate the capacitances between the same points on the x axis. Let these points be x_{a0} and x_{b0} in both cases. Then for $t > 0$ and x very large, the electric flux terminating between x_b and x_{b0} adds

$$\Delta C_b' = \frac{\epsilon}{\pi} \ln \frac{x_{b0}}{x_b} \quad (33)$$

to C_{ab}', while the flux terminating between x_a and x_{a0} adds

$$\Delta C_a' = \frac{\epsilon(x_a - x_{a0})}{s/2} \cdot \quad (34)$$

The total difference in capacitance per unit length between the values for $t > 0$ and $t = 0$ is therefore

$$\Delta C_a' + \Delta C_b' = 2\Delta C_2'(t/s), \quad (35)$$

where $\Delta C_2'(t/s)$ is the thickness-effect increment for the symmetrical parallel-plate configuration. Carrying the above steps out in detail leads to (6).